

RU-94-58  
July, 1994  
hep-th/9407176

# The String Partition Function for QCD on the Torus

ROBERT E. RUDD<sup>†</sup>

*Department of Physics and Astronomy  
Rutgers University  
Piscataway, New Jersey 08855-0849 USA*

## ABSTRACT

We study the free energy of the pure glue QCD string with a torus target space and the gauge groups  $SU(N)$  and (chiral)  $U(N)$ . It is highly constrained by a strong/weak gauge coupling duality which results in modular covariance. The string free energy is computed exactly in terms of modular forms for worldsheet genera 1 - 8. It has a surprisingly mild singularity in the weak gauge coupling/small area limit.

---

<sup>†</sup> E-mail: rerudd@physics.rutgers.edu

# 1 Introduction

The idea of a string theoretic formulation of QCD is as tantalizing today as it was twenty years ago. Despite its age and elusiveness, the promise of a description of the phenomena of strongly coupled gauge theory in term of strings is compelling. The many problems encountered in trying to implement this idea have shown that the formulation must differ substantially from the critical string. It must not include spacetime gravity, for instance. As a result, there are no promising models at this time.

The hope for a QCD string has been rejuvenated recently by significant progress understanding QCD in two dimensions. Pure glue QCD<sub>2</sub> has been solved exactly [1], and a great deal of work has gone into constructing a string theory from its large  $N$  expansion [2, 3, 4]. The string action that has resulted is a perturbed topological sigma model coupled to topological gravity [5, 6, 7, 8]. Perhaps a simpler formulation of the topological string is possible, but we are in a novel situation of having a string theory that has been shown to be equivalent to QCD, albeit in two dimensions.

The ultimate goal is to use the string theory of QCD<sub>2</sub> to construct, directly or indirectly, a QCD<sub>4</sub> string action. The direct extension of the sigma model to four dimensions would presumably yield some version of topological Yang-Mills, although this is a non-trivial, open problem. A further perturbation would be necessary to get dynamical (pure glue) QCD.

If the direct approach proves to be intractable, we may still make progress toward a QCD<sub>4</sub> string by identifying properties of the string that are independent of dimension. To that end, we will examine the QCD string partition function in detail, particularly focusing on QCD on the torus which most resembles critical string theory. The large  $N$  expansion of the QCD free energy has been given in terms of two different group theoretic sums [1, 4], the free energy for free fermions and the equivalent Jevicki-Sakita bosons [9, 10] and, of course, the free energy of the topological string itself [8]. The various formulations readily reveal different aspects of the free energy, but none is completely explicit. We will show how the structure of the Jevicki-Sakita expressions and the simple algebra of the heat kernel sum combine to let us compute the free energy efficiently. The free energy will be calculated exactly in terms of modular forms up to genus 8.

It is very rare to have an exact expression to the eighth order of string perturbation theory. In those cases where such an expression has been found, as in matrix models, it is possible to continue to all orders. In fact, there is a differential equation relating the free energy at a given order to that at lower orders. If such an equation were known for the QCD string, then there would be no need to display the horrendous expression for genus 8. But

the obvious candidates (slight generalizations of the holomorphic anomaly equation [5]) fail. The absence of boundary contributions to the (chiral)  $U(N)$  free energy suggests an even simpler structure—that a handle creation operator exists. We find something akin to one, but it couples to the infinitely many deformations of QCD. A simple equation in terms of the Kähler modulus alone would be much more powerful. It is plausible, but it has not been found. So we will list the exact expressions up to genus 8 and point out some surprising features that emerge.

## 2 QCD<sub>2</sub> and Its String Expansion

This section presents an overview of some of the salient aspects of QCD<sub>2</sub>. We give a brief review of the heat kernel partition function and Gross’s large  $N$  expansion of it. Next we discuss the relationship between  $SU(N)$  and  $U(N)$  2D Yang-Mills theory, and we show that the  $SU(N)$  partition function is easily computed from that of chiral  $U(N)$ .

Pure glue Yang-Mills theory is exactly solvable in two dimensions. This is largely due to the absence of propagating gluons, as only global degrees of freedom survive gauge fixing. There are no transverse gluons to propagate. The partition function is

$$Z = \int [\mathcal{D}A_\mu] e^{-\frac{N}{4\lambda} \int_\Sigma F_{\mu\nu} F^{\mu\nu} d^2x} \quad (2.1)$$

which may be calculated on any Riemann surface  $\Sigma$ . It only depends on the scaled gauge coupling  $\lambda$  ( $= g_{QCD}^2 N$ ), the area  $A$  of  $\Sigma$ , the topology (genus  $G$ ) of  $\Sigma$  and the gauge group. We will consider Riemann surfaces with no boundaries. The gauge group will be either  $SU(N)$  or  $U(N)$ , with a large  $N$  in order to get the string expansion in  $g_{st} = 1/N$ .

There is a remarkable solution of QCD<sub>2</sub> due to Migdal and Rusakov [1]. The heat kernel lattice action reproduces (2.1) in the continuum limit, and it has the powerful feature that it is renormalization group invariant. This permits a quick solution for the partition function

$$Z = \sum_R (\dim R)^{2-2G} e^{-\frac{\lambda A}{2N} C_2(R)} \quad (2.2)$$

where  $C_2(R)$  is the second Casimir and  $\dim R$  is the dimension of the representation  $R$ . The sum over irreducible representations may be expressed as a sum over different weights (i.e. Young tableaux). A Young tableau has  $n_k$  boxes in the  $k^{th}$  row and rows decreasing in length,  $n_1 \geq n_2 \geq \dots \geq n_N$ . For  $SU(N)$  each  $n_k$  is a non-negative integer, whereas for  $U(N)$  the weights  $n_k$  may be any integer. The additional irreducible representations are due

to the  $U(1)$  in  $U(N) \cong (SU(N) \times U(1))/Z_N$ . The Casimir is

$$C_2(R) = \begin{cases} \sum_{k=1}^N n_k(n_k + N + 1 - 2k) - \frac{n^2}{N} & \text{for } SU(N) \\ \sum_{k=1}^N n_k(n_k + N + 1 - 2k) & \text{for } U(N) \end{cases} \quad (2.3)$$

where  $n = \sum n_k$  and a particular choice is made for the  $U(1)$  charge in order to simplify the  $U(N)$  Casimir. (In general, there is an extra term  $\frac{\alpha n^2}{N}$ , where  $\alpha$  is determined by the  $U(1)$  charge  $\sqrt{\lambda'}$  in  $U(N)$ :  $\alpha = 1 - N\sqrt{\lambda'/\lambda}$ .) The dimension of  $R$  is given by

$$\dim R = \prod_{1 \leq i < j \leq N} \frac{n_i - n_j - i + j}{j - i}. \quad (2.4)$$

It is a polynomial in  $N$  of degree  $n$ ,

$$\begin{aligned} \dim R &= \frac{1}{n!} \sum_{\sigma \in S_n} N^{K_\sigma} \chi_R(\sigma) \\ &= \frac{d_R}{n!} N^n \exp \left\{ \sum_{k=1}^{\infty} (-1)^k \frac{\tilde{C}_{(k+1)}(R)}{k N^k} \right\} \end{aligned} \quad (2.5)$$

The first expression comes from the Frobenius formula for characters  $\chi_R(\sigma)$  of the representation  $R$  of the symmetric group  $S_n$  [4].  $K_\sigma$  is the number of cycles in the permutation  $\sigma$ . The second expression may be considered the large  $N$  expansion of the standard “factors over hooks” rule for the dimension (cf. [11]). The invariants  $\tilde{C}_{(k)}(R)$  are given by

$$\tilde{C}_{(k)}(R) = \sum_{j=1}^N \sum_{i=1}^{n_j} (i - j)^{k-1} \quad (2.6)$$

$\tilde{C}_{(k)}(R)$  is part of the  $k^{th}$  Casimir,  $C_k(R) = k! \tilde{C}_{(k)}(R) + \dots$ . For example,  $C_2^{U(N)}(R) = nN + 2\tilde{C}_{(2)}(R)$  and  $n = \tilde{C}_{(1)}(R)$ . The resulting expression for the partition function is

$$\begin{aligned} Z &= \sum_{n_i \geq n_{i+1}} \prod_{1 \leq i < j \leq N} \left( \frac{n_i - n_j - i + j}{j - i} \right)^{2-2G} e^{-\frac{\lambda A}{2N} \sum_k [n_k(n_k + N + 1 - 2k) - \frac{\alpha}{N} n^2]} \\ &= e^{\frac{\lambda A}{24}(N^2-1)} \left\{ \sum_{h_i > h_{i+1}} \prod_{1 \leq i < j \leq N} (h_i - h_j)^{2-2G} e^{-\frac{\lambda A}{2N} [(\sum_k h_k^2) - \frac{\alpha}{N} n^2]} \right\} \prod_{l=1}^{N-1} (l!)^{2G-2} \end{aligned} \quad (2.7)$$

where  $h_k = n_k + \frac{1}{2}(N + 1) - k$  (cf. [12]) and  $n = \sum n_k = \sum h_k$ .

The  $U(N)$  case is simpler both because of the form of the Casimir and because the sums run over all integers  $n_k$ , including the negative ones. This lets us express the  $G = 1$  partition function in terms of elliptic functions,

$$\begin{aligned} Z_{U(N)}^{G=1} &= \frac{1}{N!} e^{\frac{\lambda A}{24}(N^2-1)} \sum_{\substack{h_i \neq h_j \\ \forall i \neq j}} e^{-\frac{\lambda A}{2N} \sum_k h_k^2} \\ &= \frac{1}{N!} e^{\frac{\lambda A}{24}(N^2-1)} \left\{ \vartheta(t)^N - \binom{N}{2} \vartheta(t)^{N-2} \vartheta(2t) - \dots \right\} \end{aligned} \quad (2.8)$$

where  $t = i\lambda A/2\pi N$ . The corrections come from terms in  $\vartheta(t)^N$  with  $h_i = h_j$  for some  $i \neq j$ . The Jacobi theta function  $\vartheta$  is  $\vartheta_2$  when  $N$  is even and  $\vartheta_3$  when  $N$  is odd.

The  $SU(N)$  partition function is more complicated. Consider first the  $U(N)$  partition function sum restricted to  $SU(N)$  Young tableaux. It may be expressed in terms of the functions

$$\begin{aligned} \vartheta_{(N)}(t) &= \sum_{h=(3-N)/2}^{\infty} e^{\pi i t h^2} = \frac{1}{2} \vartheta(t) + \frac{1}{2} \sum_{h=(3-N)/2}^{(N-3)/2} e^{\pi i t h^2} \\ &= \vartheta(t) - \sum_{h=(N-1)/2}^{\infty} e^{\pi i t h^2} \end{aligned} \quad (2.9)$$

Then the intermediate partition function is

$$Z_{U/SU} = \frac{1}{(N-1)!} e^{\frac{\lambda A}{24N}(N^3-3N^2+5N-3)} \left\{ \vartheta_{(N)}(t)^{N-1} - \binom{N-1}{2} \vartheta_{(N)}(t)^{N-3} \vartheta_{(N)}(2t) - \dots \right\} \quad (2.10)$$

There is a related function,  $Z^+$ , in which the sum is restricted to Young tableaux with fewer than  $N/2$  boxes in any column.

$$Z^+ = \frac{2^{-N/2}}{(N/2)!} e^{\frac{\lambda A}{48}(N^2-1)} \left\{ \vartheta_2(t)^{N/2} - 2 \binom{N/2}{2} \vartheta_2(t)^{(N-4)/2} \vartheta_2(2t) - \dots \right\} \quad (2.11)$$

(for  $N$  even). The large  $N$  expansion of this seemingly ad hoc function will be the focus of much of what follows. Finally, to get the  $SU(N)$  partition function, the extra term in the Casimir must be included, but that is difficult. In any case, these expressions are curious,

but they are not much help. They work well for small  $N$ :

$$\begin{aligned}
Z_{U(1)}^{G=1} &= \vartheta_3(t) \\
Z_{U(2)}^{G=1} &= \frac{1}{2} e^{\frac{\lambda A}{8}} \left\{ \vartheta_2(t)^2 - \vartheta_2(2t) \right\} \\
Z_{U(3)}^{G=1} &= \frac{1}{6} e^{\frac{\lambda A}{3}} \left\{ \vartheta_3(t)^3 - 3\vartheta_3(t)\vartheta_3(2t) + 2\vartheta_3(3t) \right\} \\
Z_{U(2)/SU}^{G=1} &= \frac{1}{2} e^{\frac{\lambda A}{6}} \vartheta_2(t) \\
Z_{U(3)/SU}^{G=1} &= \frac{1}{2} e^{\frac{\lambda A}{6}} \left\{ \vartheta_3(t)^2 + 2\vartheta_3(t) - 2\vartheta_3(2t) - 1 \right\} \\
Z_{SU(2)}^{G=1} &= \frac{1}{2} e^{\frac{\lambda A}{8}} \left\{ \vartheta_3(t/2) - 1 \right\}
\end{aligned} \tag{2.12}$$

Unfortunately, (2.8) and (2.11) are not conducive to large  $N$  expansions, since it is difficult to determine if one term dominates the sum. Also, note that even the simpler  $U(N)$  partition function expressed in terms of elliptic functions is not a modular form, since each term has a different weight. We will see below that at each worldsheet genus the  $G = 1$  string free energy is almost a modular form, but with a different modulus  $\tau = \frac{N}{2}t$ . On the other hand, the modular weights of the theta functions do determine the small area behavior of the partition function. Since  $\vartheta_3(t) \sim t^{-\frac{1}{2}}$  as  $t \rightarrow 0$ , the leading term dominates the small area limit at finite  $N$ . If  $N$  is then taken to infinity, the partition function develops an essential singularity at  $t = 0$ . It is not clear from (2.8) if one term dominates in the large  $N$  limit, which should be taken first. We will see below that the string partition function has an essential singularity at  $\lambda A = 0$  which is the phase transition that occurs at finite coupling on the sphere [12].

The string expansion for  $Z$  is a large  $N$  expansion. This is explained in detail in [2] and [4], but a brief discussion of the structure will help motivate the ensuing analysis. Gross and Taylor have shown how  $SU(N)$  representations with relatively small Casimirs (of order  $N$ ) and small dimensions give the leading contribution to the partition function at large  $N$ , yielding a series that has many properties of closed string perturbation theory. A Young tableau with a small number of boxes has a relatively small Casimir, and it makes a leading contribution. But for  $SU(N)$ , a representation  $R$  and its complex conjugate  $\bar{R}$  have the same Casimir and the same dimension, so they make the same contribution to  $Z$ . This leads to a natural factorization of representations. The “chiral” representations are those with no more than  $N/2$  boxes in any column, and the “anti-chiral” representations are those

whose complex conjugate is chiral with no column of  $N/2$  boxes.<sup>1</sup> (Recall that if  $R$  has  $c_j$  boxes in its  $j^{th}$  column, then  $\bar{R}$  has  $N - c_j$  boxes in its  $j^{th}$  column from the right.) Then any representation is expressed uniquely as the Young product of an anti-chiral and a chiral representation; i.e. any tableau is a chiral tableau joined to an anti-chiral tableau.

The physical partition function may be obtained from the chiral  $U(N)$  partition function,  $Z^+$ , in which the sum is restricted to chiral  $SU(N)$  representations,

$$Z^+ = \sum_{R^+} (\dim R)^{2-2G} e^{-\frac{\lambda A}{2N} 2\tilde{C}_{(2)}} e^{-n\tilde{A}} \quad (2.13)$$

This is a well-defined function whose asymptotic expansion is the holomorphic topological string perturbation theory.  $\tilde{A} = \frac{1}{2}\lambda A$ , but it is kept formally independent of  $A$  so that the extra piece of the  $SU(N)$  Casimir may be obtained by differentiation with respect to  $\tilde{A}$ . For example, the chiral  $SU(N)$  partition function is given by

$$Z_{SU(N)}^+ = e^{(\frac{\lambda A}{2N^2})\partial_{\tilde{A}}^2} Z^+(\tilde{A}) \Big|_{\tilde{A}=\frac{1}{2}\lambda A}. \quad (2.14)$$

The free energy  $F = \log Z$  is

$$\begin{aligned} F_{SU(N)}^+ &= F^+ + \sum_{m=1}^{\infty} \tilde{P}_m \left( x_j = P_{2j}(F^{+'}, F^{+''}, \dots, F^{+(2j)})/j! \right) \left( \frac{\lambda A}{2N^2} \right)^m \\ &= F^+ + \left( \frac{\lambda A}{2N^2} \right) \left[ F^{+''} + (F^{+'})^2 \right] \\ &\quad + \frac{\left( \frac{\lambda A}{2N^2} \right)^2}{2!} \left[ F^{+(4)} + 4F^{+'}F^{+(3)} + 3(F^{+''})^2 + 6(F^{+'})^2 F^{+''} + (F^{+'})^4 \right] + \dots \end{aligned} \quad (2.15)$$

where the Schur polynomials  $P_n$  are generated by  $e^{x_k z^k} = \sum P_n(x_1, \dots, x_n) z^n$ , the polynomials  $\tilde{P}_n$  are generated by  $\log(1 + x_k z^k) = \sum \tilde{P}_n(x_1, \dots, x_n) z^n$ , and the primes denote  $\partial_{\tilde{A}}$ . The full  $SU(N)$  partition function on the torus is

$$Z_{SU(N)}^{(G=1)} = e^{(\frac{\lambda A}{2N^2})(\partial_{\tilde{A}_1} - \partial_{\tilde{A}_2})^2} Z^+(\tilde{A}_1) Z^+(\tilde{A}_2) \Big|_{\tilde{A}_1=\tilde{A}_2=\frac{1}{2}\lambda A} \quad (2.16)$$

using the formula for the Casimir of the Young product of a chiral and an anti-chiral repre-

---

<sup>1</sup>This is a slight generalization of the definition given by Gross and Taylor [4].

sentation,  $C_2(\overline{SR}) = C_2(R) + C_2(S) + 2n\tilde{n}/N$ . The  $G = 1$  free energies are related by

$$\begin{aligned}
F &= 2F^+ + \sum_{m=1}^{\infty} \tilde{P}_m \left( x_j = \frac{1}{j!} P_{2j}(0, 2F^{+''}, 0, 2F^{+(4)}, \dots, 2F^{+(2j)}) \right) \left( \frac{\lambda A}{2N^2} \right)^m \\
&= 2 \left\{ F^+ + \left( \frac{\lambda A}{2N^2} \right) F^{+''} \right. \\
&\quad + \frac{\left( \frac{\lambda A}{2N^2} \right)^2}{2!} \left[ F^{+(4)} + 6(F^{+''})^2 \right] + \frac{\left( \frac{\lambda A}{2N^2} \right)^3}{3!} \left[ F^{+(6)} + 30 F^{+''} F^{+(4)} + 60(F^{+''})^3 \right] \\
&\quad \left. + \frac{\left( \frac{\lambda A}{2N^2} \right)^4}{4!} \left[ F^{+(8)} + 56 F^{+''} F^{+(6)} + 70(F^{+(4)})^2 + 840 \left( (F^{+''})^4 + (F^{+''})^2 F^{+(4)} \right) \right] + \dots \right\}
\end{aligned} \tag{2.17}$$

Note that only even derivatives of  $F^+$  enter (2.17). As a result  $F^+$  and  $F$  (but not  $F_{SU(N)}^+$  or  $F_{U(N)}$ ) have a simple modular structure independent of the zero point energy. Once the chiral  $U(N)$  free energy is known, it is straight-forward to calculate  $F_{SU(N)}^+$  and  $F_{SU(N)}$ . Of course, it is trivial to get  $F_{U(N)}^+$  from the full  $SU(N)$  free energy,  $F_{SU(N)}$ , because of its modular structure.

The form of (2.16) suggests that for  $G = 1$  the  $U(N)$  chiral partition function is on the same footing as the more physical non-chiral  $SU(N)$  partition function. This is remarkable since a topological string theory reproducing simple Hurwitz space (i.e. the moduli space of chiral  $U(N)$ , cf. [8]) is relatively easy to construct. There are no contributions from the boundary of moduli space; the contact terms vanish. A very simple, explicit perturbation gives the full theory. This is in striking contrast to the complications of  $G \neq 1$ .

The formula for the dimension of a composite representation is not especially simple, so (2.16) and (2.17) require unwieldy corrections for  $G \neq 1$ . In fact, of all the group invariants, only the quadratic Casimir has such a simple decomposition. The higher Casimirs,  $C_k(\overline{SR})$ , decompose into a sum of products of the lower Casimirs  $C_l(R)$  and  $C_l(S)$  with  $l \leq k$ . The full partition function of QCD perturbed by  $C_k$ , may be expressed as a deformation of two copies of  $Z^+$  depending on the couplings of all the lower  $C_j$ 's. The higher the Casimir, the more couplings that must be differentiated. The dimension of  $R$  may be expressed in terms of the  $C_j$ 's as well, but in the large  $N$  limit it takes infinitely many, so the analog of (2.16) for  $G \neq 1$  requires derivatives with respect to infinitely many couplings.

The perturbed chiral  $U(N)$  partition function may be written

$$Z^+(\tilde{A}, \tilde{A}_2, \dots) = \sum_{R^+} (\dim R)^{2-2G} \exp \left( \sum_k \tilde{A}_k C_k(R)/N^{k-1} \right) e^{-n\tilde{A}}. \tag{2.18}$$

This partition function results from the chiral reduction of a renormalization group invariant heat kernel lattice action as before, but it leads to perturbations of the Yang-Mills action



$F^2$  by higher powers of the field strength  $(F^2)^k$  in the continuum. Its large  $N$  expansion is string-like, since only even powers of  $1/N$  arise. This results from a cancellation in the sum over representations. It follows from the definition (2.6) that  $\tilde{C}_{(k)}(R) = (-1)^{k-1} \tilde{C}_{(k)}(S)$ , if the columns of  $R$ 's Young tableau are the rows of  $S$ 's. Since the Casimir  $C_k(R)$  is a homogeneous polynomial in  $\tilde{C}_{(l)}(R)/N^{l-1}$ , the relative minus sign cancels any occurrence of an odd power of  $1/N$  arising from the perturbations or the dimension (2.5). This argument immediately extends to the  $SU(N)$  case and the full partition function. Of course, there are other properties a string perturbation expansion should possess, but these will be verified elsewhere [13].

The large  $N$  chiral partition function for  $U(N)$  Yang-Mills theory on the torus has been shown to be the following sum over classes of  $n$ -sheeted holomorphic covering maps  $\nu_n$  [4]

$$F^+ = \sum_{g=G}^{\infty} N^{2-2g} \sum_{n=1}^{\infty} \sum_{[\nu_n]: \Sigma_g \rightarrow \Sigma_{G=1}} f_{g,\nu_n}^G \left(\frac{1}{2}\lambda A\right)^i e^{-\frac{1}{2}\lambda n A} \quad (2.19)$$

where  $\Sigma_g$  ( $\Sigma_G$ ) is the genus  $g$  ( $G$ ) worldsheet (space-time) Riemann surface, and  $[\nu_n]$  is a class of  $n$ -sheeted covers with  $i$  square root branch points.  $f_{g,\nu_n}^G$  is the number of holomorphic maps divided by a symmetry factor characterizing the moduli space of maps. The maps satisfy the Riemann-Hurwitz relation

$$2(g-1) = 2n(G-1) + i. \quad (2.20)$$

Since  $i = 2g - 2$  for maps to the torus, the factors multiplying the exponential in (2.19) combine into  $(\lambda A/2N)^{2g-2}$ . The QCD partition function on other ( $G \neq 1$ ) Riemann surfaces is more complicated than (2.19). Even the  $SU(N)$  partition function on the torus is more complicated, with contributions coming from the boundary of moduli space where worldsheet handles are collapsed to a point in space-time [3]. For  $G \neq 1$  at  $\lambda A = 0$ , the free energy is a sum of the orbifold Euler characteristics of the moduli spaces [8].

We will study QCD on the torus, so there is no contribution to the free energy coming from genus 0 worldsheets. The lowest genus worldsheet to contribute is an unbranched single cover of the target space, so  $g_{min} = G$  according to the Riemann-Hurwitz formula. Since  $F_0 = 0$  the relationship between the free energy and the partition function is simple:

$$\begin{aligned} Z &= e^{F_1} \exp \left\{ \sum_{g=2}^{\infty} (1/N)^{2g-2} F_g \right\} \\ &= e^{F_1} \left[ (1/N)^2 F_2 + (1/N)^4 \left( F_3 + \frac{1}{2} F_2^2 \right) + \cdots \right]. \end{aligned} \quad (2.21)$$

We will usually discuss the free energy below, but the partition function is very similar in form. For instance, when  $F_g^+$  has modular weight  $6g-6$ , so does  $Z_g^+$  apart from the factor  $e^{F_1^+} = \eta$ . This eta function factor gives the partition function an essential singularity at zero area/coupling which can cause difficulties (like order of limits problems) that do not arise with the free energy.

### 3 QCD On The Torus

In this section we will proceed to study some properties of the  $G = 1$  free energy in detail. The free energy on the torus has an interesting structure reminiscent of critical string theory, but not shared with QCD on other Riemann surfaces. It transforms nicely under target space modular transformations, as expected for a string theory. It is not exactly modular covariant, but it turns out that  $F_g^+$  is an anomalous modular form of weight  $6g-6$ . The deviations from modular covariance are of an interesting form, and they conspire to give an unexpectedly mild behavior at small area and/or weak gauge coupling. This may be related to a hidden symmetry in the string theory. The section begins with the derivation of a generating function for  $F_g^+$ , followed by a general description of its modular properties. Then both are used to calculate  $F_g^+$  and  $F_g$  for  $g = 1, \dots, 8$ , exactly in terms of modular forms. Finally, we examine the large and small area behavior of the free energy.

QCD on the torus is simpler in many ways than on the sphere or on higher genus surfaces. The dimension factor drops out of the heat kernel expression for the partition function (2.2):

$$Z^+ = \sum_{n_1 \geq n_2 \geq \dots \geq n_{N/2} \geq 0} e^{-\frac{\lambda \tilde{A}}{2N} \sum_k [n_k(n_k+1-2k)]} e^{-n\tilde{A}}. \quad (3.22)$$

In the Gross-Taylor description this means that no omega points or omega-inverse points are required. The chiral partition function just counts simply branched maps. Despite the absence of omega points, the free energy is non-trivial since the map  $\nu : \Sigma_g \rightarrow \Sigma_{G=1}$  can wrap arbitrarily many times. The wrapping number  $n$  drops out of the Riemann-Hurwitz formula (2.20) for  $G = 1$ . This allows non-trivial modular transformations of the Kähler modulus  $\tilde{A}$ . In addition to formulations of QCD<sub>2</sub> using gauge theory and string theory techniques that work for any Riemann surface, QCD on the torus has been reformulated as a two dimensional free fermion theory and a Jevicki-Sakita boson theory [9, 10]. The relative simplicity of QCD on the torus along with the alternative formulations allows us to say a great deal about the string theory.

The partition function (3.22) is similar to a theta function, as we saw above in (2.8). It

would be nice to extract the large  $N$  expansion directly from the theta functions, but that has proved to be difficult. We will now develop a way to sum the series (3.22) that is more amenable to a large  $N$  expansion. Since the sum is restricted to chiral representations, the factor of  $1/N$  appearing in the first exponential in (3.22) is the string coupling.  $Z^+$  may be expressed in terms of a generating function:

$$\begin{aligned} Z^+ &= \sum_{n_1 \geq n_2 \geq \dots \geq n_{N/2} \geq 0} e^{-\frac{\lambda A}{2N} \sum_k [(-\partial_{\beta_k})(-\partial_{\beta_k} + 1 - 2k)]} e^{-\sum_k \beta_k n_k} \Big|_{\beta_j = \tilde{A}} \\ &= \sum_{m_j \geq 0, \forall j} e^{-\frac{\lambda A}{2N} \sum_k (k \partial_{\alpha_k}^2 + k^2 \partial_{\alpha_k} + 2k \partial_{\alpha_k} \sum_{l=k+1}^{\infty} \partial_{\alpha_l})} e^{-\sum_k \alpha_k m_k} \Big|_{\alpha_j = j \tilde{A}} \end{aligned} \quad (3.23)$$

where  $m_k = n_k - n_{k+1}$  (i.e.  $n_k = \sum_{l=k}^{\infty} m_l$ ),  $\beta_k = \alpha_l - \alpha_{l-1}$  and  $\partial_{\beta_k} = \sum_{l=k}^{\infty} \partial_{\alpha_l}$ . All of the sums over the  $m_j$  begin at zero, so the generating function is just a product of geometric sums. Summing these series (and allowing for a zero point energy  $\epsilon_0$ ) we find

$$\begin{aligned} Z^+ &= e^{\epsilon_0 \tilde{A}/24} \exp \left\{ -\frac{\lambda A}{2N} \sum_{k=1}^{\infty} \left( k \partial_{\alpha_k}^2 + k^2 \partial_{\alpha_k} + 2k \partial_{\alpha_k} \sum_{l=k+1}^{\infty} \partial_{\alpha_l} \right) \right\} \frac{1}{\tilde{\eta}(\alpha)} \Big|_{\alpha_j = j \tilde{A}} \\ &= e^{\epsilon_0 \tilde{A}/24} \exp \left\{ -\frac{\lambda A}{2N} \sum_{k=1}^{\infty} \left( k^2 \partial_{\alpha_k} + \partial_{\tilde{A}} \partial_{\alpha_k} - \frac{1}{2} \sum_{l=1}^{\infty} |k-l| \partial_{\alpha_k} \partial_{\alpha_l} \right) \right\} \frac{1}{\tilde{\eta}(\alpha)} \Big|_{\alpha_j = j \tilde{A}} \end{aligned} \quad (3.24)$$

where the generalized eta function is given by

$$\frac{1}{\tilde{\eta}(\alpha)} = e^{\tilde{A}/24} \prod_{j=1}^{\infty} \frac{1}{1 - e^{-\alpha_j}} + \dots \quad (3.25)$$

The omitted terms correct for the fact that we have allowed an infinite number of rows instead of stopping  $j$  at  $N/2$ , the maximum for a chiral tableau. These terms are proportional to some power of  $e^{-N}$ , so they are exponentially small non-perturbative corrections. We can ignore them. When (3.25) is evaluated at  $\alpha_j = j \tilde{A}$  in the large  $N$  limit, the generalized eta function becomes the usual eta function

$$\tilde{\eta}|_{\alpha_j = j \tilde{A}} \xrightarrow{N \rightarrow \infty} \eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (3.26)$$

with  $q = e^{-\tilde{A}}$  (so that the Kähler modulus of the torus is  $\tau = -\tilde{A}/(2\pi i)$ ). This is the genus  $g = 1$  chiral partition function for  $SU(N)$  and  $U(N)$  Yang-Mills on the torus found in [2].

The free energy is the “connected” part of (3.24). It may be computed using the Campbell-Baker-Hausdorff formula. This is particularly simple because  $\log \eta(\alpha) = -\frac{\tilde{A}}{24} + \sum_j \log(1 - e^{\alpha_j})$ , and different derivatives cannot act on the same term in the sum.

It is key for a closed string formulation of QCD<sub>2</sub> that the odd powers of  $1/N$  vanish in the partition function. This leads to some non-trivial identities[14]. For example, the term in (3.24) proportional to  $1/N$  (corresponding to genus  $g = 3/2$ ) is

$$\sum_{k=1}^{\infty} \left\{ -\frac{k(k-1)q^k}{1-q^k} + \frac{2kq^{2k}}{1-q^{2k}} + \sum_{l=1}^{\infty} \frac{(k+l)q^{k+l}}{(1-q^k)(1-q^l)} \right\} = 0. \quad (3.27)$$

The argument given above for the absence of these terms, based on Gross's original discussion, is essentially group theoretic. It would be interesting to understand the identity (3.27) and its generalizations from the point of view of number theory.

The free energy is almost, but not quite, invariant under modular transformations. Already we have seen that  $F_1^+ = -\log \eta$ . This is not modular invariant, contrary to what one would expect for a string theory. Under the transformation of the Kähler modulus  $\tilde{A} \rightarrow 4\pi^2/\tilde{A}$ , the free energy has a modular anomaly:  $\log \eta(4\pi^2/\tilde{A}) = \log \eta(\tilde{A}) + \frac{1}{2} \log(\tilde{A}/2\pi)$ . It becomes modular invariant if we make an ansatz that the theory has a holomorphic anomaly.<sup>2</sup> The anomaly makes a contribution to  $F^+$  that depends on  $\tilde{\tilde{A}}$  as well as  $\tilde{A}$ . Then

$$F_1^+ = -\log \left\{ \sqrt{\frac{1}{2}(\tilde{A} + \tilde{\tilde{A}})} \eta(\tilde{A}) \eta(\tilde{\tilde{A}}) \right\}. \quad (3.28)$$

This is modular invariant, and the original free energy is recovered by sending  $\tilde{\tilde{A}} \rightarrow \infty$ , up to an infinite shift in the zero point energy. Modular invariance is an extremely useful property. It is a kind of strong/weak coupling duality in the gauge coupling  $\lambda$ , and it determines the general structure of the free energy in terms of modular forms.

The higher genus free energies are also expected to be modular covariant after the non-holomorphic completion. The string coupling transforms non-trivially (with weight  $-1$ ), so the higher genus free energies should transform with a definite modular weight. Douglas [9] was able to show that this is the case using his formulation of QCD<sub>2</sub> on the torus in terms of a Jevicki-Sakita two dimensional boson system:

$$Z^+ = \int \mathcal{D}\varphi e^{-\int \partial\varphi \bar{\partial}\varphi + g_{st} A(\partial\varphi)^3}. \quad (3.29)$$

The propagator is  $\langle \partial\varphi(z) \partial\varphi(0) \rangle = -\partial^2 \log \vartheta_1(z|\tilde{A}) - \frac{4\pi^2}{\tilde{A} + \tilde{\tilde{A}}} = \wp(z|\tilde{A}) + \pi^2 E_2(\tilde{A})/3 - \frac{4\pi^2}{\tilde{A} + \tilde{\tilde{A}}}$ , where  $\vartheta_1$  is the Jacobi theta function and the Weierstrass  $\wp$  function is given by  $\wp(z|\tilde{A}) = z^{-2} + \sum_k 2(2k-1) \zeta(2k) E_{2k}(\tilde{A}) z^{2k-2}$ . The functions  $E_{2k}$  are the Eisenstein series—weight  $2k$  modular forms. They are described below. The point is that the propagator has modular

---

<sup>2</sup>The holomorphic anomaly arises in topological string theory, coming from the contribution of BRST exact operators (with anti-holomorphic couplings) at the boundary of moduli space. [5]

weight two, where a weight  $k$  modular form transforms as  $M_k(-1/\tau) = \tau^k M_k(\tau)$ . This determines the weight of the genus  $g$  free energy

$$F_g^+ = \frac{1}{(2g-2)!} \left( \frac{\lambda A}{2N} \right)^{2g-2} \left\langle \left[ \int : (\partial\varphi)^3 : \right]^{2g-2} \right\rangle_{connected}. \quad (3.30)$$

It has  $3g-3$  propagators, giving a total weight  $6g-6$ .

It would be fantastic if we could do the integrals to determine which weight  $6g-6$  modular form  $F_g^+$  is. Unfortunately, these integrals are very difficult. Douglas has done the  $F_2^+$  integral [15], and we have done the  $F_3^+$  integrals. The higher genus integrals are extremely difficult, since the convolution of offset  $\wp$  functions is not elliptic. Also, as the genus  $g$  increases, there are a growing number of diagrams and integrals. In fact, the number of diagrams at genus  $g$  is proportional to  $g!$ , as easily seen in the zero dimensional  $\varphi^3$  integral. In addition, ever higher weight modular forms entering from the  $\wp$  function must be reduced. It does not seem promising to calculate the higher genus free energies this way. Fortunately, the fact that they are modular forms determines them up to a few coefficients. These coefficients may be found by taking parametric derivatives of the generalized  $\eta$  function (3.24).

A set of  $3g^2/4$  ( $(3g^2 + 2g - 5)/4$  for odd  $g$ ) modular forms comprises a basis at weight  $6g-6$ . The basis is made of products of  $E_2, E_4$  and  $E_6$  or  $E_2, E_2'$  and  $E_2''$ , which have weight 2, 4 and 6, respectively. Most texts on modular forms write the basis in terms of  $E_4$  and  $E_6$ , but for our purposes the second basis is preferable. The two bases are interchangeable, since  $E_4 = (E_2)^2 + 12E_2'$  and  $E_6 = E_2^3 + 18E_2 E_2' + 36E_2''$ . Also, note that  $E_2''' = \frac{3}{2}[E_2']^2 - E_2 E_2''$  and  $E_{k+2} = E_k E_2 + \frac{12}{k} E_k'$  for  $k = 4, 6, \dots$ . The prime means differentiation with respect to  $\tilde{A}$ , so  $f' = \partial_{\tilde{A}} f = \frac{-1}{2\pi i} \partial_\tau f$ , and for finite  $\tilde{A}$  it becomes the covariant derivative

$$D_{\tilde{A}} = \partial_{\tilde{A}} + \frac{k}{\tilde{A} + \bar{\tilde{A}}}. \quad (3.31)$$

acting on a weight  $k$  modular form. Regardless of which basis is used, the genus  $g$  free energy is determined up to roughly  $3g^2/4$  coefficients

$$F_g^+ = \left( \frac{\lambda A}{2N} \right)^{2g-2} \sum_{k=0}^{3g-3} \sum_{2l+3m=3g-3-k} c_{kl} E_2^k (E_2')^l (E_2'')^m. \quad (3.32)$$

The coefficients  $c_{kl}$  are rational numbers to be determined.

The  $k^{th}$  Eisenstein series,  $E_k$ , is

$$\begin{aligned} E_k &= \frac{1}{2\zeta(k)} \sum'_{m,n} \frac{1}{(m\tau + n)^k} \\ &= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned} \quad (3.33)$$

for  $k \in 2\mathbf{Z}^+$  (cf. [16]). The number theoretic function  $\sigma_k(n)$  is the sum of the  $k^{th}$  power of the divisors of  $n$ ,  $\sigma_k(n) = \sum_{d|n} d^k$ , and  $B_k$  is the  $k^{th}$  Bernoulli number. Every  $E_k$  is a modular form of weight  $k$ , except  $E_2$ . There is no modular form of weight two, although  $E_2$  comes close. The Eisenstein series transform as

$$E_2(-1/\tau) = \tau^2 \left( E_2(\tau) + \frac{12}{2\pi i \tau} \right) \quad E_k(-1/\tau) = \tau^k E_k(\tau) \quad k = 4, 6, \dots \quad (3.34)$$

The fact that the higher Eisenstein series are modular forms is easily seen from the definition (3.33) since the sums converge absolutely (and uniformly) on the upper half plane. Since  $E_2$  has a modular anomaly, it must be covariantized by adding a term depending on  $\tilde{A}$

$$E_2 = 24 \partial_{\tilde{A}} F_1^+ = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n - \frac{12}{\tilde{A} + \tilde{A}} \quad (3.35)$$

so  $E_2 = 1 - 24e^{-\tilde{A}} - 72e^{-2\tilde{A}} - \dots$ .

The  $G = 1$  free energy for chiral  $U(N)$  (restricted to  $SU(N)$  tableaux) has been calculated exactly up to worldsheet genus 8. It would be possible to continue up to genus 11, but beyond that the computation takes too long even on fast computers. A glance at the genus 8 result (A.56) reveals that these expressions would fill pages. The first few free energies are given by

$$F_1^+ = \frac{\epsilon_0}{24} \tilde{A} - \log \eta \quad (3.36)$$

$$\begin{aligned} F_2^+ &= \frac{\left(\frac{-\lambda A}{2N}\right)^2}{2! 2^4 3^4 5} \{5E_2^3 - 3E_2 E_4 - 2E_6\} \\ &= -\frac{\left(\frac{-\lambda A}{2N}\right)^2}{2! 90} \{E_2 E_2' + E_2''\} \end{aligned} \quad (3.37)$$

and

$$\begin{aligned} F_3^+ &= \frac{\left(\frac{-\lambda A}{2N}\right)^4}{4! 2^7 3^6} \{-6E_2^6 + 15E_2^4 E_4 + 4E_2^3 E_6 - 12E_2^2 E_4^2 - 12E_2 E_4 E_6 + 7E_4^3 + 4E_6^2\} \\ &= \frac{\left(\frac{-\lambda A}{2N}\right)^4}{4! 54} \{7[E_2']^3 + 3[E_2'']^2\} \end{aligned} \quad (3.38)$$

$F_1^+$  and  $F_2^+$  have been calculated previously by Gross [2] and by Douglas [15], respectively. The higher free energies were not known. The remaining expressions  $F_4^+, \dots, F_8^+$  are increasingly lengthy, so they are listed in the appendix.

Using (2.15) and (2.17) these results for the chiral  $U(N)$  free energies may be converted into the more complicated chiral  $SU(N)$  and full  $SU(N)$  free energies. The additional terms in the full  $SU(N)$  free energy coming from turning off the  $U(1)$  coupling and combining the chiral sectors are still modular forms (up to the  $E_2$  modular anomaly), but they have lower weights. The corrections for the chiral  $SU(N)$  free energy do not have a definite weight because  $F^{+'} = (E_2 - 1)/24$  enters (2.15). The same is true of the non-chiral  $U(N)$  free energy. The general form of the full  $SU(N)$  free energy (the analog of (3.32) for chiral  $U(N)$ ) is

$$F_g = g_{st}^{2g-2} \sum_{l=0}^{g-1} (\lambda A)^{2g-2-l} \sum_{k=0}^{3g-3-l} E_2^k M_{6g-6-2l-2k,l} \quad (3.39)$$

where  $M_{k,l}$  is a true modular form of weight  $k$  or a weight  $k$  combination of  $E_2'$  and  $E_2''$  in the other basis. For example,

$$\begin{aligned} F_3 = & \frac{\left(\frac{-\lambda A}{2N}\right)^4}{4! 27} \left\{ 7 [E_2']^3 + 3 [E_2'']^2 \right\} - \frac{\left(\frac{-\lambda A}{2N}\right)^2 \left(\frac{\lambda A}{2N^2}\right)}{4! 3} 4 E_2' E_2'' \\ & + \frac{\left(\frac{\lambda A}{2N^2}\right)^2}{4! 4} \left\{ 4 E_2 E_2'' - 5 [E_2']^2 \right\} \end{aligned} \quad (3.40)$$

$F_1$  through  $F_4$  are given in the appendix. The higher  $SU(N)$  free energies are omitted for brevity.

A surprise about the free energy is that it has an unexpectedly mild singularity as  $\lambda A \rightarrow 0$ . To see this it will be useful to know the large and small area behavior of the Eisenstein series. The series are normalized so that

$$E_k \rightarrow 1 + \mathcal{O}(e^{-\tilde{A}}) \quad \text{as } \tilde{A} \rightarrow +\infty \quad (3.41)$$

for  $k = 2, 4, \dots$ . The small area limit is related to this by a modular transformation

$$\begin{aligned} E_k & \rightarrow \tau^{-k} + \mathcal{O}(e^{-2\pi i/\tau}) \quad \text{as } \tau \rightarrow i0^+ \quad (k = 4, 6, \dots) \\ & \rightarrow (-\tilde{A}/2\pi i)^{-k} + \mathcal{O}(e^{-4\pi^2/\tilde{A}}) \quad \text{as } \tilde{A} \rightarrow 0^+ \\ E_2 & \rightarrow \frac{1}{\tau^2} - \frac{12}{2\pi i \tau} + \mathcal{O}(e^{-2\pi i/\tau}) \quad \text{as } \tau \rightarrow i0^+ \\ & \rightarrow \frac{-4\pi^2}{\tilde{A}^2} + \frac{12}{\tilde{A}} + \mathcal{O}(e^{-4\pi^2/\tilde{A}}) \quad \text{as } \tilde{A} \rightarrow 0^+ \end{aligned} \quad (3.42)$$

It is amusing to note that these formulas also result from using the Euler-Maclaurin formula on the  $q$ -expansion (3.33). The usual derivation of the modular transformation laws is quite different.

Consider the expression for the partition function (3.24). In the large area limit  $F_g^+$  goes like  $e^{-2\tilde{A}}$  for  $g \geq 2$ . There is no constant term, which would come from non-covering maps. Similarly there is no single cover term proportional to  $e^{-\tilde{A}}$ , since higher genus surfaces must be at least double covers of the torus. The absence of the constant term shows up in the expressions for  $F_g^+$  since each term contains a factor of  $E_2'$  or  $E_2''$  which begin with  $e^{-\tilde{A}}$ . It is part of the reason that the expressions are simpler in this basis. The small  $\tilde{A}$  limit is much more interesting. A weight  $6g - 6$  modular form would go like  $(1/\tilde{A})^{6g-6}$  as  $\tilde{A} \rightarrow 0^+$ . A much softer singularity is observed. Up to terms of order  $\mathcal{O}(e^{-1/\tilde{A}})$  the following expressions hold:

$$F_2^+ = \left(-\frac{\lambda A}{2N}\right)^2 \left\{ \frac{2}{3\tilde{A}^3} - \frac{2\pi^2}{3\tilde{A}^4} + \frac{8\pi^4}{45\tilde{A}^5} \right\} \quad (3.43)$$

$$F_3^+ = \left(-\frac{\lambda A}{2N}\right)^4 \left\{ \frac{-8}{\tilde{A}^6} + \frac{16\pi^2}{\tilde{A}^7} - \frac{100\pi^4}{9\tilde{A}^8} + \frac{224\pi^6}{81\tilde{A}^9} \right\} \quad (3.44)$$

$$F_4^+ = \left(-\frac{\lambda A}{2N}\right)^6 \left\{ \frac{2272}{9\tilde{A}^9} - \frac{2272\pi^2}{3\tilde{A}^{10}} + \frac{8096\pi^4}{9\tilde{A}^{11}} - \frac{41504\pi^6}{81\tilde{A}^{12}} + \frac{48256\pi^8}{405\tilde{A}^{13}} \right\} \quad (3.45)$$

$$F_5^+ = \left(-\frac{\lambda A}{2N}\right)^8 \left\{ \frac{-13504}{\tilde{A}^{12}} + \frac{54016\pi^2}{\tilde{A}^{13}} - \frac{834304\pi^4}{9\tilde{A}^{14}} + \frac{7010816\pi^6}{81\tilde{A}^{15}} - \frac{17887904\pi^8}{405\tilde{A}^{16}} + \frac{11958784\pi^{10}}{1215\tilde{A}^{17}} \right\} \quad (3.46)$$

$$F_6^+ = \left(-\frac{\lambda A}{2N}\right)^{10} \left\{ \frac{15465472}{15\tilde{A}^{15}} - \frac{15465472\pi^2}{3\tilde{A}^{16}} + \frac{105156608\pi^4}{9\tilde{A}^{17}} - \frac{418657280\pi^6}{27\tilde{A}^{18}} + \frac{572409344\pi^8}{45\tilde{A}^{19}} - \frac{2467804672\pi^{10}}{405\tilde{A}^{20}} + \frac{33778284544\pi^{12}}{25515\tilde{A}^{21}} \right\} \quad (3.47)$$

The expressions for  $F_7^+$  and  $F_8^+$  are easily calculated as well from (A.55) and (A.56).

In general the small area behavior of the free energy is

$$F_g^+ = \left(-\frac{\lambda A}{2N}\right)^{2g-2} \sum_{k=3g-3}^{4g-3} \frac{c_{k,g}\pi^{2(k-3g+3)}}{\tilde{A}^k} + \mathcal{O}(e^{-1/\tilde{A}}) \quad (3.48)$$

where each  $c_{k,g}$  is a rational number. This form may be proven using the Campbell-Baker-Hausdorff formula to get  $F_g^+$  from  $Z^+$  (3.24), and then using the Euler-Maclaurin formula



to replace the sums with integrals. This small area behavior is interesting for a number of reasons. It might be the result of a symmetry in  $\text{QCD}_2$ , or one in the underlying string theory. If it is, the symmetry would be novel. The small area limit is also interesting because of its implications for an equation relating the free energy at a given genus to that at lower genera. If the free energy satisfies a string master equation like the holomorphic anomaly equation, it must satisfy it as  $\tilde{A} \rightarrow 0$ . It is very easy to check that no simple equation will work. Of course, the equation could be more complicated. For example, if the full  $SU(N)$  partition function satisfies the simple holomorphic anomaly equation, then the chiral  $U(N)$  partition function will satisfy a non-polynomial (in  $1/N$ ) differential equation resulting from (3.24).

## 4 Conclusions

In the recent program to extract a string theory from Migdal's explicit solution of  $\text{QCD}_2$ , we have taken a step backward from the starting point, in a sense. We have computed even more explicit expressions for the  $\text{QCD}_2$  string free energy up to genus 8. These calculations relied on a strong/weak gauge coupling duality that is exact at each order of string perturbation theory (but is violated by the non-perturbative corrections). The modular structure of the free energy is familiar from topological string theory, but there does not seem to be a simple holomorphic anomaly equation for  $F^+$ .

It might be expected that there would exist a handle generating operator since  $F^+$  receives no contribution coming from the boundary of moduli space (collapsed handles or tubes). In some sense the differential operator generating the partition function from the generalized eta function (3.24) plays this role. It is not as simple as one would like, since it couples to each row number separately. This is equivalent to having it couple to the infinitely many deformations of  $\text{QCD}_2$  (the higher Casimir perturbations), rather than coupling to the Kähler modulus  $\tilde{A}$  alone. So it is an important open question in the worldsheet theory to understand how the free energy at a given genus is related to that at lower genera.

Another interesting question we have raised is the cause of the softening of the  $\lambda A \rightarrow 0$  singularity. We proved the property by looking directly at the small area limit of the heat kernel expression for the partition function. It would be very interesting to have a worldsheet explanation for this effect (although it might just be accidental).

In any case, the exact expressions for the free energy offer many possibilities for further investigations. The goal of the recent work on two dimensional Yang-Mills theory is to make

progress toward understanding four dimensional QCD, or at least to learn more about string theories without spacetime gravity. We have exhibited properties of the two dimensional free energy that could have a bearing on either of these two interesting goals.

Acknowledgements:

I wish to thank Robbert Dijkgraaf, Mike Douglas, David Gross, Greg Moore and Ken Intriligator for many interesting discussions. Also, many thanks to the Aspen Center for Physics, where some of this work was done.

## A Appendix: The Free Energy up to Genus 8

The  $U(N)$  free energy on the torus restricted to one chiral sector is

$$F_1^+ = -\frac{\epsilon_0}{24} \tilde{A} - \log \eta \quad (\text{A.49})$$

$$F_2^+ = -\frac{\left(\frac{-\lambda A}{2N}\right)^2}{2! 90} \{E_2 E_2' + E_2''\} \quad (\text{A.50})$$

$$F_3^+ = \frac{\left(\frac{-\lambda A}{2N}\right)^4}{4! 54} \{7 [E_2']^3 + 3 [E_2'']^2\} \quad (\text{A.51})$$

$$\begin{aligned} F_4^+ = \frac{\left(\frac{-\lambda A}{2N}\right)^6}{6! 54} \{ & 27 E_2 [E_2']^4 - 36 [E_2]^2 [E_2']^2 E_2'' - 746 [E_2']^3 E_2'' \\ & + 12 [E_2]^3 [E_2'']^2 + 246 E_2 E_2' [E_2'']^2 - 106 [E_2'']^3 \} \end{aligned} \quad (\text{A.52})$$

$$\begin{aligned} F_5^+ = \frac{\left(\frac{-\lambda A}{2N}\right)^8}{8! 81} \{ & 162 [E_2]^4 [E_2']^4 + 5940 [E_2]^2 [E_2']^5 + 85299 [E_2']^6 + 19821 [E_2'']^4 \\ & - 216 [E_2]^5 [E_2']^2 E_2'' - 8352 [E_2]^3 [E_2']^3 E_2'' - 165388 E_2 [E_2']^4 E_2'' \\ & + 72 [E_2]^6 [E_2'']^2 + 2928 [E_2]^4 E_2' [E_2'']^2 + 97980 [E_2]^2 [E_2']^2 [E_2'']^2 \\ & + 291334 [E_2']^3 [E_2'']^2 - 16896 [E_2]^3 [E_2'']^3 - 116472 [E_2] E_2' [E_2'']^3 \} \end{aligned} \quad (\text{A.53})$$

$$\begin{aligned}
F_6^+ = & \frac{\left(\frac{-\lambda A}{2N}\right)^{10}}{10! \, 27} \left\{ 216 [E_2]^7 [E_2']^4 + 12420 [E_2]^5 [E_2']^5 + 935442 [E_2]^3 [E_2']^6 \right. \\
& + 9054978 E_2 [E_2']^7 - 288 [E_2]^8 [E_2']^2 E_2'' - 17136 [E_2]^6 [E_2']^3 E_2'' \\
& - 1798164 [E_2]^4 [E_2']^4 E_2'' - 21957340 [E_2]^2 [E_2']^5 E_2'' - 49812944 [E_2]^6 [E_2']^6 E_2'' \\
& + 96 [E_2]^9 [E_2'']^2 + 5904 [E_2]^7 E_2' [E_2'']^2 + 1129560 [E_2]^5 [E_2']^2 [E_2'']^2 \\
& + 16998104 [E_2]^3 [E_2']^3 [E_2'']^2 + 85070724 [E_2] [E_2']^4 [E_2'']^2 - 230768 [E_2]^6 [E_2'']^3 \\
& - 4242768 [E_2]^4 E_2' [E_2'']^3 - 43083696 [E_2]^2 [E_2']^2 [E_2'']^3 - 55574424 [E_2']^3 [E_2'']^3 \\
& \left. + 5750160 [E_2]^3 [E_2'']^4 + 22892460 E_2 E_2' [E_2'']^4 - 2132916 [E_2'']^5 \right\}
\end{aligned} \tag{A.54}$$

$$\begin{aligned}
F_7^+ = & \frac{\left(\frac{-\lambda A}{2N}\right)^{12}}{12! \, 81} \left\{ 2592 [E_2]^{10} [E_2']^4 + 204768 [E_2]^8 [E_2']^5 + 89976744 [E_2]^6 [E_2']^6 \right. \\
& + 1909213524 [E_2]^4 [E_2']^7 - 3456 [E_2]^{11} [E_2']^2 E_2'' - 279936 [E_2]^9 [E_2']^3 E_2'' \\
& - 177698880 [E_2]^7 [E_2']^4 E_2'' - 4174329312 [E_2]^5 [E_2']^5 E_2'' \\
& - 62394892148 [E_2]^3 [E_2']^6 E_2'' + 1152 [E_2]^{12} [E_2'']^2 \\
& + 95616 [E_2]^{10} E_2' [E_2'']^2 + 116614944 [E_2]^8 [E_2']^2 [E_2'']^2 \\
& + 3013923216 [E_2]^6 [E_2']^3 [E_2'']^2 + 63439748328 [E_2]^4 [E_2']^4 [E_2'']^2 \\
& - 25422720 [E_2]^9 [E_2'']^3 - 719490240 [E_2]^7 E_2' [E_2'']^3 \\
& - 26470959696 [E_2]^5 [E_2']^2 [E_2'']^3 - 216898783824 [E_2]^3 [E_2']^3 [E_2'']^3 \\
& + 3707598864 [E_2]^6 [E_2'']^4 + 44962956000 [E_2]^4 E_2' [E_2'']^4 \\
& - 27342815040 [E_2]^3 [E_2'']^5 + 21435613473 [E_2]^2 [E_2']^8 \\
& + 354642271752 [E_2]^2 [E_2']^5 [E_2'']^2 + 246384220752 [E_2]^2 [E_2']^2 [E_2'']^4 \\
& - 221060144076 E_2 [E_2']^7 E_2'' - 540228415584 E_2 [E_2']^4 [E_2'']^3 \\
& - 75540784608 E_2 E_2' [E_2'']^5 + 37289952912 [E_2']^9 \\
& + 343299239380 [E_2']^6 [E_2'']^2 + 186490620756 [E_2']^3 [E_2'']^4 \\
& \left. + 4465217052 [E_2'']^6 \right\}
\end{aligned} \tag{A.55}$$

$$\begin{aligned}
F_8^+ = & \frac{\left(\frac{-\lambda A}{2N}\right)^{14}}{14! \, 243} \left\{ 31104 [E_2]^{13} [E_2']^4 + 3146688 [E_2]^{11} [E_2']^5 + 9381744000 [E_2]^9 [E_2']^6 \right. \\
& + 284491245600 [E_2]^7 [E_2']^7 - 41472 [E_2]^{14} [E_2']^2 E_2'' - 4278528 [E_2]^{12} [E_2']^3 E_2'' \\
& - 27941270346864 [E_2]^6 [E_2']^6 E_2'' - 606326798592 [E_2]^8 [E_2']^5 E_2'' \\
& - 18712973376 [E_2]^{10} [E_2']^4 E_2'' + 13824 [E_2]^{15} [E_2'']^2 + 1453824 [E_2]^{13} E_2' [E_2'']^2 \\
& + 12436238592 [E_2]^{11} [E_2']^2 [E_2'']^2 + 428976125568 [E_2]^9 [E_2']^3 [E_2'']^2 \\
& + 27464636483952 [E_2]^7 [E_2']^4 [E_2'']^2 - 2753687808 [E_2]^{12} [E_2'']^3 \\
& - 100795663872 [E_2]^{10} E_2' [E_2'']^3 - 11670697945728 [E_2]^8 [E_2']^2 [E_2'']^3 \\
& - 191631253050048 [E_2]^6 [E_2']^3 [E_2'']^3 + 1796319251712 [E_2]^9 [E_2'']^4 \\
& + 35809340742912 [E_2]^7 E_2' [E_2'']^4 - 65393019413760 [E_2]^6 [E_2'']^5 \\
& + 10411506055692 [E_2]^5 [E_2']^8 + 374290947363144 [E_2]^5 [E_2']^5 [E_2'']^2 \\
& + 622325551111920 [E_2]^5 [E_2']^2 [E_2'']^4 - 314884801430312 [E_2]^4 [E_2']^7 E_2'' \\
& - 1915319874231576 [E_2]^4 [E_2']^4 [E_2'']^3 - 599386224758688 [E_2]^4 E_2' [E_2'']^5 \\
& + 95567800131858 [E_2]^3 [E_2']^9 + 2553509520593988 [E_2]^3 [E_2']^6 [E_2'']^2 \\
& + 3327996287536944 [E_2]^3 [E_2']^3 [E_2'']^4 + 207585191777040 [E_2]^3 [E_2'']^6 \\
& - 1483200593325666 [E_2]^2 [E_2']^8 E_2'' - 6326492920454424 [E_2]^2 [E_2']^5 [E_2'']^3 \\
& - 2174062501379952 [E_2]^2 [E_2']^2 [E_2'']^5 + 287886256181076 E_2 [E_2']^{10} \\
& + 4807697809119300 E_2 [E_2']^7 [E_2'']^2 + 5189221094011812 E_2 [E_2']^4 [E_2'']^4 \\
& + 436733614643256 E_2 E_2' [E_2'']^6 - 1172778987525768 [E_2']^9 E_2'' \\
& - 3523420607226032 [E_2']^6 [E_2'']^3 - 1116320466888120 [E_2']^3 [E_2'']^5 \\
& \left. - 17870538853512 [E_2'']^7 \right\}
\end{aligned} \tag{A.56}$$

The full  $SU(N)$  free energy on the torus is

$$F_1 = \frac{\epsilon_0}{12} \left( \frac{\lambda A}{2} \right) - 2 \log \eta \quad (\text{A.57})$$

$$F_2 = -\frac{\left(\frac{-\lambda A}{2N}\right)^2}{2! 45} \{E_2 E_2' + E_2''\} + \frac{\left(\frac{\lambda A}{2N^2}\right)}{2! 6} E_2' \quad (\text{A.58})$$

$$\begin{aligned} F_3 = & \frac{\left(\frac{-\lambda A}{2N}\right)^4}{4! 27} \{7 [E_2']^3 + 3 [E_2'']^2\} - \frac{\left(\frac{-\lambda A}{2N}\right)^2 \left(\frac{\lambda A}{2N^2}\right)}{4! 3} \{4 E_2' E_2''\} \\ & + \frac{\left(\frac{\lambda A}{2N^2}\right)^2}{4! 4} \{4 E_2 E_2'' - 5 [E_2']^2\} \end{aligned} \quad (\text{A.59})$$

$$\begin{aligned} F_4 = & \frac{\left(\frac{-\lambda A}{2N}\right)^6}{6! 27} \{27 E_2 [E_2']^4 - 36 [E_2]^2 [E_2']^2 E_2'' - 746 [E_2']^3 E_2'' \\ & + 12 [E_2]^3 [E_2'']^2 + 246 E_2 E_2' [E_2'']^2 - 106 [E_2'']^3\} \\ & + \frac{\left(\frac{-\lambda A}{2N}\right)^4 \left(\frac{\lambda A}{2N^2}\right)}{6! 3} \{150 [E_2']^4 - 160 E_2 [E_2']^2 E_2'' + 40 [E_2]^2 [E_2'']^2 + 180 E_2' [E_2'']^2\} \\ & + \frac{\left(\frac{-\lambda A}{2N}\right)^2 \left(\frac{\lambda A}{2N^2}\right)^2}{6!} \{30 E_2 [E_2']^3 - 20 [E_2]^2 E_2' E_2'' - 120 [E_2']^2 E_2'' + 60 E_2 [E_2'']^2\} \\ & + \frac{\left(\frac{\lambda A}{2N^2}\right)^3}{6! 24} \{65 [E_2']^3 - 360 [E_2]^2 [E_2']^2 + 240 [E_2]^3 E_2'' + 420 E_2 E_2' E_2'' - 480 [E_2'']^2\} \end{aligned} \quad (\text{A.60})$$

The free energy for genus 5, 6, 7 and 8 may be computed from (A.53) to (A.56), but they are omitted to save space.

## References

- [1] A.A. Migdal, *Zh. Eksp. Teor. Fiz.* **69** (1975) 810 (JETP **42** (1976) 413);  
B.Y. Rusakov, *Mod. Phys. Lett.* **A5** (1990) 693;  
E. Witten, *Comm. Math. Phys.* **141** (1991) 153
- [2] D.J. Gross, *Nucl. Phys.* **B400** (1993) 161
- [3] J.A. Minahan, *Phys. Rev.* **D47** (1993) 3430
- [4] D.J. Gross and W. Taylor, *Nucl. Phys.* **B400** (1993) 181; *Nucl. Phys.* **B403** (1993) 395

- [5] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Nucl. Phys.* **B405** (1993) 279; “Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes,” Harvard preprint HUTP-93/A025, Sept. 93, hep-th/9309140
- [6] R. Dijkgraaf and R.E. Rudd, unpublished
- [7] P. Hořava, “Topological Strings and QCD in Two Dimensions,” Cargese talk, Chicago preprint EFI-93-66, hep-th/9311156
- [8] S. Cordes, G. Moore, S. Ramgoolam, “Large N 2D Yang-Mills Theory and Topological String Theory,” Rutgers preprint RU-94-20, hep-th/9402107
- [9] M.R. Douglas, “Conformal Field Theory Techniques for Large N Group Theory”, Rutgers preprint RU-93-13, hep-th/9303159; “Some QCD2 Calculations”, Rutgers preprint RU-93-13, May 1993
- [10] J.A. Minahan and A.P. Polychronakos, *Phys. Lett.* **B312** (1993) 155; *Phys. Lett.* **B326** (1994) 288
- [11] H. Georgi, *Lie Algebras in Particle Physics* (Benjamin/Cummings Publishing, Menlo Park, 1982) p. 120;  
A.O. Barut and R. Rączka, *Theory of Group Representations and Applications* (World Scientific, Singapore, 1986)
- [12] M.R. Douglas and V. Kazakov, *Phys. Lett.* **B319** (1993) 219
- [13] R.E. Rudd, in preparation
- [14] D.J. Gross, private communication
- [15] M.R. Douglas, “Conformal Field Theory Techniques in Large N Yang-Mills Theory”, Cargese Lectures, hep-th/9311130
- [16] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, 2cd ed. (Springer-Verlag, New York, 1993)